# Model theory of Hardy fields

# Abstract

These are notes from Lou van den Dries' lectures on the model theory of Hardy fields, during the Fields thematic semester on tame geometry, January-June 2022.

The notes are incomplete, and start with considerations on second order linear equations over Hardy fields. In particular, the material on Hausdorff and Hardy fields and semialgebraic differential equations of order 1 over Hardy fields is not here. So until I include it (which I may never do), the notes here start from page 13.

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**02-17: Lecture 10**

**Notation 3.0.1.** *For* $f, g \in \mathcal{C}$ *with* $f, g > 0$*, we write* $f \gg g$ *if* $f \succ \max(g, g^{-1})^{\mathbb{N}}$*, and* $f \ggcurly g$ *if* $\log f \succ \log g$*. We also write* $f \gtrapprox g$ *if* $f \not\gg g$ *and* $g \not\gg f$ *and* $f \succeq g$ *if* $\log f \succcurlyeq \log g$*.*

**Proposition 3.0.2.** *Let* $H$ *be a Hardy field, let* $h \in H$ *with* $h \preccurlyeq x$*. Then there is a unique* $y \in \mathcal{C}^2$ *with*

$$y'' + y = h \tag{3.0.1}$$

*and* $y - f$ *is non-oscilllating for all* $f \in H$*.*

So $y$ generates a Hardy field over $H$.

**Lemma 3.0.3.** *Suppose that* $h \in \mathcal{C}_a^1$ *and* $h' \geqslant C \in \mathbb{R}^>$*. Then for all* $y \in \mathcal{C}_a^1$ *with* $y'' + y = h$*, we have* $y > \mathbb{R}$*.*

**Proof.** We have $h > 0$ since $h' \geqslant C > 0$. Let $a \in \mathbb{R}$ with $h(t) > 0$ and $h'(t) \geqslant C$ for all $t \geqslant a$. It is enough to show the result for the solution

$$y(t) = \int_a^t \sin(t - s)\, h(s)\, \mathrm{ds} = \int_a^t \sin(s)\, h(t - s)\, \mathrm{ds}$$

of (3.0.1). Set $t_n = a + 2\pi n$ for all $n \in \mathbb{N}$. So for $t \geqslant t_n$, we have

$$\int_0^{2n\pi} \sin(s)\, h(t - s)\, \mathrm{ds} = \sum_{i=0}^{n} \int_{2i\pi}^{2(i+1)\pi} \sin(s)\, h(t - s)\, \mathrm{ds}$$

where for $i \in \{0, \ldots, n\}$, we have

$$\begin{aligned} \int_{2i\pi}^{2(i+1)\pi} \sin(s)\, h(t - s)\, \mathrm{ds} &= \int_0^{\pi} \sin(s)\, [h(t - 2\, i\, \pi - s) - h(t - 2\,(i + 1)\, \pi - s)]\, \mathrm{ds} \\ &\geqslant \int_0^{\pi} \sin(s)\, \pi\, C\, \mathrm{ds} \qquad \text{since } h'(u) \geqslant C \text{ on } [a, +\infty) \\ &= 2\, \pi\, C. \end{aligned}$$

It follows that $\int_0^{2n\pi} \sin(s)\, h(t - s)\, \mathrm{ds} \geqslant 2\, n\, \pi\, C$ for $t \geqslant t_n$. In particular $y(t_n) \geqslant 2\, n\, \pi\, C$. Routine computations show that between $t_n$ and $t_{n+1}$, we also have $y(t) \geqslant 2\, n\, \pi\, C$, so $y > \mathbb{R}$ as a germ. □

**Lemma 3.0.4.** *Let* $h$ *be Hardian with* $|h| \gg x$*, and let* $y \in \mathcal{C}^2$ *with* $y'' + y = h$*. Then* $y \gg x$*.*

**Proof.** Changing $h$ with $-h$ and $y$ with $-y$ if necessary, we can arrange that $h > 0$. For $n \in \mathbb{N}$, set $y_n := y - x^n$. We have $y_n'' + y_n = h_n := h - Q(x)$ for some polynomial $Q(x) = -n\,(n - 1)\, x^{n-2} + \cdots$ with integer coefficients. We still have $h_n \succ x^{\mathbb{N}}$, whence $h_n' \succ x^{\mathbb{N}}$ since $h$ is Hardian. In particular $h_n$ satisfies the conditions of Lemma 3.0.3, whence $y_n > \mathbb{R}$. In particular $y \succ x^{\mathbb{N}}$. □

**Theorem 3.0.5.** *Let $H$ be a real-closed Hardy field and let $h \in H$. Then either*

*I. $f'' + f - h \preccurlyeq x^n$ for some $f \in H$ and $n \in \mathbb{N}$, or*

*II. $f'' + f - h \succ x^n$ for all $f \in H$ and $n \in \mathbb{N}$.*

*In case I, there is a unique $y \in \mathcal{C}^2$ with $y'' + y = h$ which generates a Hardy field $H(y, y')$ over H. In case II, every $y \in \mathcal{C}^2$ with $y'' + y = h$ generates a Hardy field $H(y, y')$ over H.*

**Proof.** Suppose that we are in case $I$, with corresponding $(f, n)$. Setting $z = y - f$, the ODE

$$y'' + y = h$$

transforms into

$$z'' + z = h - (f'' + f) \preccurlyeq x^n$$

Then Proposition 3.0.2 gives the result. Suppose now that we are in case $II$. Let $y \in \mathcal{C}^2$ with $y'' + y = h$. We will show that $y - f$ is non-oscillating for all $f \in H$. For $f \in H$, we have

$$(y - f)'' + (y - f) = h - (f'' + f) \gg x$$

so Lemma 3.0.4 gives that$y - f \gg x$, whence in particular $y$ is non-oscillating. Therefore $y$ generates a Hardy field over $H$. □

**Remark 3.0.6.** Suppose we are in case $I$ of the previous theorem. Then the unique corresponding $y$ actually generates a Hardy field over any Hardy field extension $H^* \supseteq H$. Indeed the $f, n$ of case $I$ for $H$ also witness case $I$ for $H^*$, whence by unicity, we are still in case $I$ in $H^*$ with the same $y$.

Suppose we are in case $II$ of the theorem. Then there are continuum-many solutions $y$ witnessing case $II$, any two of which are incompatible (because of the oscillating nature of non-zero solutions of the homogeneous harmonic equation). Plausible: in case II, for any two solutions $y_1, y_2$, the fields $H(y_1, y_1')$ and $H(y_2, y_2')$ are isomorphic.

Does case $II$ actually occur? Boshernitzan show that this is the case in $H = \mathbb{R}(x, \mathrm{e}^{x^2})^{\mathrm{rc}}$ for $y'' + y = \mathrm{e}^{x^2}$. One line in Boshernitzan's proof should be made clearer. Indeed Boshernitzan uses the following fact about complex linear differential equations:

**Fact** Let $f_1, \ldots, f_n, g$ be holomorphic functions on a non-empty simply connected region $\Omega$ in $\mathbb{C}$. Then any solution $y$ of

$$y^{(n)} + f_1\, y^{(n-1)} + \cdots + f_n\, y = g$$

which is holomorphic on a non-empty open $U \subseteq \Omega$ extends holomorphicaly to $\Omega$. So come back to

$$y'' + y = \mathrm{e}^{x^2},$$

we note that $x \mapsto \mathrm{e}^{x^2}$ is entire, so any solution $y$ extends into an entire function. This closes the gap in Boshernitzan's proof (see upcoming notes from Lou).

Let $H$ be a Hardy field. A germ $y \in \mathcal{C}^{<\infty}$ is said *Hardian over* $H$, or $H$*-Hardian* if $y$ lies in some Hardy field extension of $H$. Boshernitzan defines a larger Hardy field

$$E(H) \ := \ \{f \in \mathcal{C}^{<\infty} : y \text{ is } H^*\text{-Hardian for all } H^* \text{ Hardy field extension of } H\}.$$

Equivalently, this is the intersection of all maximal Hardy fields containing $H$.

**What do we know about $\boldsymbol{E(H)}$?** We have $E(E(H)) = E(H)$ by definition. Since any maximal Hardy field is Liouville-closed, the field $E(H)$ is Liouville-closed. We also have $\mathbb{R} \subseteq E(H)$ and $\cos(h), \sin(h) \in E(H)$ for all $h \in E(H)^{\preccurlyeq 1}$ by previous results.

Let us focus on $E := E(\mathbb{Q})$, which is the set of "most Hardian germs", i.e. germs that are contained in all maximal Hardy fields. By Boshernitzan's result, the ODE

$$y'' + y = \mathrm{e}^{x^2}$$

has no solution in $E$. The differential field $E$ is differentially algebraic (over $\mathbb{Q}$ say). We'll give a sketch of proof shortly. As a consequence, no $y \in E$ is transexponential or sublogarithmic. Moreover $E$ is closed under composition [3, Theorem 6.8].

Fernando Sanz suggests looking at the differential equation

$$y'' + y = \frac{\mathrm{e}^x}{x},$$

whose solutions are supposed to be definable in *o*-minimal structures.

**Question 1.** Does any element of $E^{>\mathbb{R}}$ have a level?

**Answer 1.** The answer is positive and can be deduced from a result by van der Hoeven. Consider the field $\mathbb{T}_g$ of grid-based transseries. Let $\mathbb{T}_{\mathrm{da}}$ denote the subfield of $\mathbb{T}_g$ of d-algebraic grid-based transseries over $\mathbb{R}$. This is da-closed in $\mathbb{T}_g$, hence is a model of the elementary theory $T_{\mathrm{LE}}$ of log-exp transseries as an ordered valued differential field. By [5, Theorem 5.12], there is a Hardy field $\mathcal{H}$ closed under exp and log and an isomorphism $(\mathbb{T}_{\mathrm{da}}, +, \times, <, \prec, \partial, \log) \longrightarrow (H, +, \times, <, \prec, {}', \log)$. In particular $\mathcal{H}$ is a model of $T_{\mathrm{LE}}$. Let $f \in E$ and assume for contradiction that $f \notin \mathcal{H}$. Let $M \supseteq \mathcal{H}$ be a maximal Hardy field. We have $f \in M$ by definition of $E$. Now $f \in M \setminus \mathcal{H}$ must be $d$-transcendant over $\mathcal{H}$, hence also over $\mathbb{R}$. This contradicts Boshernitzan's result that each element of $E$ is d-algebraic. Thus $E \subseteq \mathcal{H}$. In particular, the field $E$ embeds into $\mathbb{T}_g$ as an ordered exponential field, so each $f \in E^{>\mathbb{R}}$ has a level $n \in \mathbb{Z}$.

**Question 2.** Is $E$ closed under compositional inversion (of positive infinite germs)?

**Question 3.** Is $E$ contained in $\mathcal{H}_{\mathcal{R}}$ for some o-minimal expansion $\mathcal{R} = (\mathbb{R}, +, \times, <, \ldots)$ of the real ordered field? More precisely, do we have $E \subseteq \mathrm{Pfaff}(\mathbb{R}_{\mathrm{alg}})$ (Pfaffian closureas per [7])?

**Sketch of proof that $E$ is differentially algebraic.** Suppose $f \in \mathcal{C}^{<\infty}$ is Hardian but not differentially algebraic. One (Boshernitzan, for instance) can show that for any sufficiently small $\phi \in (\mathcal{C}^{<\infty})^{\times}$ i.e. if $\phi^{(n)} \prec \frac{1}{\exp_{\mathbb{N}}}$ for all $n \in \mathbb{N}$, then $f + \phi \sin$ is also Hardian. But then there is a maximal Hardy field containing $f + \phi \sin$, which does not contain $f$: a contradiction. □

**General fact.** *[in a paper of Lou and Matthias]Let $H$ be a Hardy field containing $x$. Let $P \in H[Y, Y', \ldots, Y^{(n)}]$. Then there is an $f \in H^{>}$ with either*

$$P(y, y', \ldots, y^{(n)}) > 0$$

*for all $H$-Hardian germs $y > \exp_{\mathbb{N}}(f)$, or*

$$P(y, y', \ldots, y^{(n)}) < 0$$

*for all $y > \exp_{\mathbb{N}}(f)$.*

**02-22: Lecture 11**

# Chapter 4
# Cuts and differential algebra

## 4.1 Asymptotic couples

Let $\mathcal{H}$ be a Hardy field, let $v: \mathcal{H}^{\times} \longrightarrow \Gamma$ be the natural valuation of $\mathcal{H}$ seen as an ordered field. We write $\mathcal{O}_{\mathcal{H}}$ , or sometimes $\mathcal{H}^{\preccurlyeq}$, for the corresponding valuation ring. Recall that for non-zero $f, g \not\asymp 1$, we have $f \asymp g \Longrightarrow f' \asymp g'$. So we have an operation $'$ on the value group

$$\begin{array}{rcl} \Gamma^{\neq} & \longrightarrow & \Gamma \\ v f & \longmapsto & (v f)' := v f'. \end{array}$$

We also have a function

$$\begin{array}{rcl} \Gamma^{\neq} & \longrightarrow & \Gamma \\ v f & \longmapsto & (v f)^{\dagger} := (v f)' - (v f), \end{array}$$

which will have useful properties. This function and the structure $(\Gamma, {}^{\dagger})$ were introduced by M. Rosenlicht. The dagger operation is in particular a valuation on the ordered group $\Gamma$, that is, for $\alpha, \beta \in \Gamma^{\neq}$ with $\alpha + \beta \neq 0$, we have $(\alpha + \beta)^{\dagger} \geqslant \min(\alpha^{\dagger}, \beta^{\dagger})$. The properties of $\dagger$ on $H$ showed earlier also imply that for $\alpha \in \Gamma^{\neq}$ and $\beta > 0$, we have $\alpha^{\dagger} < \beta'$ and $\beta' > 0$.

### 4.1.1 Asymptotic couples

**Definition 4.1.1.** [6, 2] *An* **asymptotic couple** *is a pair* $(\Gamma, \psi)$ *where* $\Gamma$ *is a linearly ordered Abelian group and a function* $\psi: \Gamma^{\neq} \longrightarrow \Gamma$*, such that for all* $\alpha, \beta \in \Gamma^{\neq}$*, we have*

- **AC1.** *If* $\alpha + \beta \neq 0$*, then* $\psi(\alpha + \beta) \geqslant \min(\psi\alpha, \psi\beta)$.
- **AC2.** $\psi(k\alpha) = \psi(\alpha)$ *for all* $k \in \mathbb{Z} \setminus \{0\}$.
- **AC3.** *If* $\alpha > 0$*, then* $\alpha + \psi\alpha > \psi\beta$.

*If in addition, we have* $0 < \alpha < \beta \Longrightarrow \psi\alpha \geqslant \psi\beta$*, then we call* $(\Gamma, \psi)$ *and* $H$*-asymptotic couple. We will often write* $\alpha^{\dagger} := \psi\alpha$ *and* $\alpha' := \psi\alpha + \alpha$ *for all* $\alpha \in \Gamma^{\neq}$*. We say that* $(\Gamma, \psi)$ *has small derivation if* $\alpha' > 0$ *for all* $\alpha > 0$.

So the value group of $\mathcal{H}$ with the dagger operation defined above is an $H$-asymptotic couple. In fact the same holds if $\mathcal{H}$ is any H-field with small derivation. It is sometimes convenient to extend $\psi$ to a function $\psi: \Gamma \longrightarrow \Gamma \sqcup \{\infty\}$ by setting $\Gamma < \infty$ and $\psi 0 := \infty$. This preserves the axioms **AC1** — **AC3** above.

The basic facts about asymptotic couples and $H$-asymptotic couples were either derived by Rosenlicht or proved in [2, Sections 6.5, 9.1 and 9.2].

**Basic facts** Let $(\Gamma, \dagger)$ be an asymptotic couple, and let $\alpha, \beta, \gamma \in \Gamma$. We have the following.

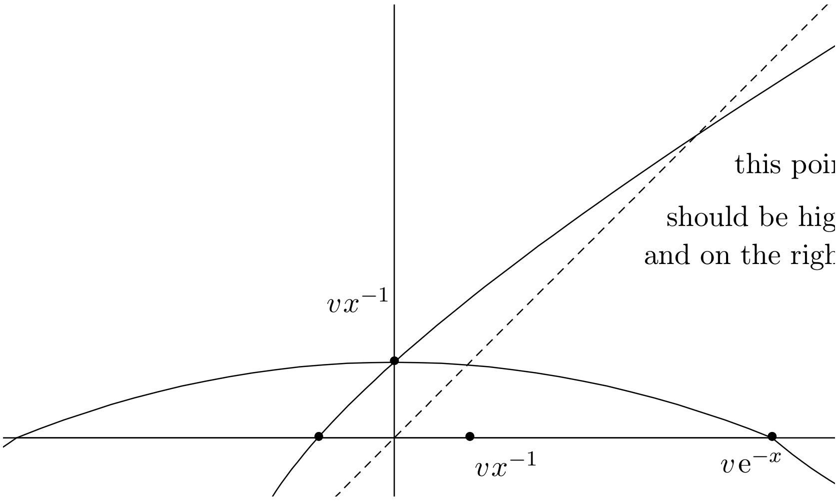
i. If $\alpha, \beta \neq 0$ and $\alpha \neq \beta$, then $\alpha^{\dagger} - \beta^{\dagger} = o(\alpha - \beta)$, i.e. $\mathbb{N}^{>} (\alpha^{\dagger} - \beta^{\dagger}) < \alpha - \beta$.

ii. The function $\Gamma^{\neq} \longrightarrow \Gamma; \gamma \mapsto \gamma'$ is strictly increasing.

iii. If $\alpha, \beta \neq 0$, then $\mathbb{N}^{>}(\alpha^{\dagger} - \beta^{\dagger}) < \max(\alpha, -\alpha)$.

A consequence of iii is that $\psi$ extends uniquely to the divisible hull $\mathbb{Q}\Gamma$ of $\Gamma$ in such a way that the corresponding structure $(\mathbb{Q}\Gamma, \psi)$ is an asymptotic couple.

A general intuition about asymptotic couples can be summurized in the following graph of $\psi: \Gamma \longrightarrow \Gamma$ on the asymtptoc couple:




On this graph, we see that for all $\alpha \in \Gamma^{>}$ and $\beta \in \Gamma$ we have $\alpha' > \beta^{\dagger}$. We also see that if $\Psi \cap \Gamma^{>}$ is non-empty, then we have $\alpha > 0 \Longrightarrow \alpha' > 0$ for all $\alpha$, and $\psi$ has a unique fixed point which we suggestively denote $vx^{-1}$ here, and sometimes call 1.

**Definition 4.1.2.** *An asymptotic field is a valued differential field $(K, \preccurlyeq, \partial)$ such that for all non-zero $f, g \prec 1$, we have $f \prec g \Longleftrightarrow f' \prec g'$.*

Then the value group $vK^{\times}$ of $K$ gives rise to an asymtotic couple defined as in the case of Hardy fields or H-fields, (which are particular cases of asymptotic fields). It follows from the defintion that for all non-zero $f, g \not\asymp 1$, we have $f \prec g \Longleftrightarrow f' \prec g'$. In this context, or for general asymptotic couples, we define

$$\Psi := \psi(\Gamma^{>}) = \psi(\Gamma^{\neq}).$$

We have $\Psi < (\Gamma^{>})'$ in $(\Gamma, <)$. If $\mathcal{H}$ is a Liouville-closed Hardy field, then the set $\Psi$ is downward closed (i.e. initial in $(\Gamma, <)$), because each derivative is a logarithmic derivative.

### 4.1.2 Further basic facts about couples

See [2, Sections 6.5, 9.1 and 9.2] for proofs of the following facts, some of which were already proved by Rosenlicht.

The set $\Gamma \setminus (\Gamma^{\neq})'$ has at most one element, and this element equals $\max \Psi$ if $\Psi$ has a maximum.

**Corollary 4.1.3.** *There is at most one $\gamma \in \Gamma$ with $\Psi < \gamma < (\Gamma^{>})'$.*

Such an element $\gamma$ is called a *gap*, and there cannot be gaps if $\Psi$ has a maximum. Gaps remains gaps when taking divisible hulls. We call $(\Gamma, \psi)$ *grounded* if $\Psi$ has a maximum. We say that $(\Gamma, \psi)$ has *asymptotic integration* if $\Gamma = (\Gamma^{\neq})'$.

**Trichotomy for $H$-asymptotic couples** We have the following trichotomy, given an $H$-asymptotic couple $(\Gamma, \psi)$. Exactly one of the following occurs:

I. $(\Gamma, \psi)$ has a gap.

II. $(\Gamma, \psi)$ is grounded.

III. $(\Gamma, \psi)$ has asymptotic integration.

**Example 4.1.4.** Here are examples for each case of the trichotomy.

I. Note that for $\Gamma = \{0\}$, the element 0 is a gap. It is not trivial to construct other $H$-asymptotic couples with a gap, but they can be realized as non-Archimedean Hardy fields or fields of transseries.

II. Suppose $\Gamma$ is finitely generated as an Abelian group, or more generally that its rational rank (i.e. $\dim_{\mathbb{Q}}(\mathbb{Q}\,\Gamma)$) is finite, or even the rank of $\Gamma$ has a valued group. Then $(\Gamma, \psi)$ is grounded, since in fact $\Psi$ is finite.

III. If $\mathcal{H}$ is a Hardy field which is closed under integration. Then its asymptotic couple has asymptotic integration.

Consider normalized $H$-asymptotic couples $(\Gamma, \psi, 1)$ where $1 > 0$ and $\psi(1) = 1$, and let $T_{\text{Ha}}$ denote the corresponding first-order theory.

**Theorem 4.1.5.** [1] *A normalized $H$-asymptotic couple $(\Gamma, \psi, 1)$ is existenially closed with respect to $T_{\text{Ha}}$ if and only if all the following conditions are satisfied:*

*i.* $\Gamma$ *is divisible.*

*ii.* $(\Gamma, \psi)$ *has asymptotic integration.*

*iii.* $\Psi$ *is initial in* $(\Gamma, <)$.

*In other words, we have a model companion for $T_{\text{Ha}}$. Moreover, this model companion has QE in the extended language with a unary predicate for $\Psi$.*

This is in particular the case for Liouville-closed Hardy fields (or Liouville-closed H-fields with small derivation). In that case the group $\Gamma$ is naturally an ordered vector space over $\mathbb{R}$, using exp and log to define real powers

$$f^r := \exp(r \log f), r \in \mathbb{R}$$

of strictly positive elements. We also have a QE result for two sorted structure expanded with this scalar multiplication. Moreover, this last structure is interpretable in the ambient Hardy field / H-field. One interprets $(r, v f) \mapsto r\, v f$ using the differential equation

$$y^{\dagger} f = r,$$

whose ambiguities are absorbed by the valuation.

## 4.2 Cuts in Hardy fields

(continued in **03-01: Lecture 12**) Extending a Hardy field $\mathcal{H}$ means in particular realizing cuts in $\mathcal{H}$. We will focus on cuts in $\mathcal{H}^{>}$ for convenience, i.e. on subsets of $\mathcal{H}^{>}$ without supremum in $\mathcal{H}^{>}$. We assume that $\mathbb{R} \subsetneq \mathcal{H}$. There are five particularly important cuts:

| Symbol | Definition | Realization | In $\mathbb{R}(x)$ | In $\mathrm{Li}(\mathbb{R})$ |
|---|---|---|---|---|
| $\infty$ | $\mathcal{H}^{>}$ | $y > \mathcal{H}$ | $\mathrm{e}^x$ | $\exp_\omega$ |
| $\ell_\infty$ | $\mathcal{H}^{>,\preccurlyeq}$ | $\mathbb{N} < y < \mathcal{H}^{>\mathbb{N}}$ | $\log x$ | $\log_\omega$ |
| $\boldsymbol{\gamma}$ | $\{f : \int f \preccurlyeq 1\}$ | $(\mathcal{H}^{>,\preccurlyeq})' < y < (\mathcal{H}^{>\mathbb{R}})^\dagger$ | $\log x, \frac{1}{\log x}$ | $\log_\omega', \left(\frac{1}{\log_\omega}\right)'$ |
| $\boldsymbol{\lambda}$ | $(-\boldsymbol{\gamma}^\dagger)$ | $-(\mathcal{H}^{>\mathbb{R}})^{\dagger\dagger} < y < -(\mathcal{H}^{\prec,>})'^\dagger$ | $\frac{1}{x} + \frac{1}{x \log x}$ | $\frac{1}{x} + \frac{1}{x \log x} + \cdots$ |
| $\boldsymbol{\omega}$ | $\omega(\boldsymbol{\lambda})$ | $\omega(\boldsymbol{\lambda}_L) < y < \omega(f) + (-f^\dagger)^2, f \in \boldsymbol{\lambda}_L$ | $\frac{1}{x^2} + \frac{1}{x^2 (\log x)^2}$ | $\frac{1}{x^2} + \frac{1}{x^2 (\log x)^2} + \cdots$ |

Note that all those cuts are definable in a uniform way in $(H, +, \times, <, ')$. We write $\boldsymbol{c} \in \mathcal{H}$ to mean that the cut $\boldsymbol{c} = (A, B)$ is realized in $\mathcal{H}$, meaning that there is an $h \in \mathcal{H}$ with $A < h < B$. We write $\boldsymbol{c} \notin \mathcal{H}$ to mean that $\boldsymbol{c}$ is realized in a Hardy field extension of $\mathcal{H}$ but not in $\mathcal{H}$. We say that $\mathcal{H}$ is $\boldsymbol{\lambda}$-free if $\boldsymbol{\lambda} \notin \mathcal{H}$. Likewise $\mathcal{H}$ is $\boldsymbol{\omega}$-free if $\boldsymbol{\omega} \notin \mathcal{H}$.

For the lambda cut and the omega cut, we have other explicit quasi-quadratic definitions, assuming that $\mathcal{H}$ is also ungrounded (so not $\mathbb{R}(x)$). In particular, the field $\mathcal{H}$ is $\boldsymbol{\lambda}$-free if and only if

$$\forall f, (\exists g, (g \succ 1, f - g^{\dagger\dagger} \succcurlyeq g^\dagger)).$$

For any differential field $(F, ')$, we have a function

$$\omega(z) := -2 z' - z^2$$

on $F$. If $F$ is an H-field, then $\omega$ is strictly increasing on $\boldsymbol{\lambda}_L$.

As long as $\mathcal{H}$ is ungrounded, there is a sequence $(\ell_\rho)_{\rho<\alpha} \in \mathcal{H}^{>\mathbb{R}}$ which is strictly decreasing, coinitial in $\mathcal{H}^{>\mathbb{R}}$ and satisfies $\ell_\rho^\dagger \succ \ell_\sigma^\dagger$ whenever $\rho < \sigma < \alpha$. The ordinal $\alpha$ is an infinite limit ordinal. In transseries, we *must* have $\alpha = \omega$ and we *can* take $\ell_n := \log_n x$ for all $n < \omega$. Then a realization of $\boldsymbol{\lambda}$ is an $\mathcal{H}$-Hardian $y$ with $\left(-\frac{1}{\ell_\rho}\right)' < y < \ell_\rho^\dagger$ for all $\rho < \alpha$. This does not depend on the choice of $(\ell_\rho)_{\rho<\alpha}$. We write $\gamma_\rho := \ell_\rho^\dagger$ as in the finite case. Writing $\lambda_\rho := -\gamma_\rho^\dagger$, then a realization of $\boldsymbol{\lambda}$ is an $\mathcal{H}$-Hardian pseudo limit of $(\lambda_\rho)_{\rho<\alpha}$. Note that for such a realization $y$, the germ $y^\dagger$ realizes $\boldsymbol{\lambda}$. So $\boldsymbol{\lambda}$-freeness implies $\boldsymbol{\gamma}$-freeness (but not the other way around). Writing $\omega_\rho := \omega(\lambda_\rho)$, we have a strictly increasing pseudo-Cauchy sequence $(\omega_\rho)_{\rho<\alpha}$, and $\mathcal{H}$ is $\boldsymbol{\omega}$-free if and only if $(\omega_\rho)_{\rho<\alpha}$ has no pseudo-limit in $\mathcal{H}$. Likewise $\boldsymbol{\omega}$-freeness implies $\boldsymbol{\lambda}$-freeness.

**Question 4.** Let $f, g \in \mathcal{C}^{<\infty}$ be Hardian and strictly positive and assume that $\int f \preccurlyeq 1$ and $\int g > \mathbb{R}$. Then must we have

$$f \prec g\ ?$$

## 4.3 Cuts and H-fields

(continued in **03-03: Lecture 13**) In order to make sense of the $\boldsymbol{\lambda}$-cut and the $\boldsymbol{\omega}$-cut, it is necessary to consider a first-order generalization of Hardy fields, i.e. the notion of H-fields (more precisely those with constant field $\mathbb{R}$).

### 4.3.1 H-fields

**Definition 4.3.1.** *Let $C$ be an ordered field. An* **H-field** *with constant field $C$ is an ordered valued differential field $(K, +, \times, <, \prec, \partial)$ with constant field $C$ such that*

**HF1.** *$f' > 0$ for all $f \in K$ with $f > C$.*

**HF2.** *We have $K^{\preccurlyeq} = C + K^{\prec}$.*

*We say that $K$ has* **small derivation** *if moreover*

**HF3.** $f' \prec 1$ *for all* $f \in K^{\prec}$.

We'll take most our H-fields to have small derivation. In particular, each H-field is an asymptotic field whose asymptotic couple is $H$-asymptotic.

### 4.3.2 Cuts in H-fields

Let us fix an ungrounded H-field $K$, let $(\Gamma, \dagger)$ denote its asymptotic couple. Then for $h \in K^{>}$, we have

$$h \in \boldsymbol{\gamma}_L \quad \text{if and only if} \quad \int h \preccurlyeq 1 \text{ in any H-field extension of } K \text{ containing } \int h,$$

$$h \in \boldsymbol{\gamma}_R \quad \text{if and only if} \quad \int h \succ 1 \text{ in any H-field extension of } K \text{ containing } \int h,$$

$$\boldsymbol{\gamma}_L < h < \boldsymbol{\gamma}_R \quad \text{if and only if} \quad \int h \preccurlyeq 1 \text{ and } \int h \succ 1 \text{ in two distinct H-field extensions of } K.$$

This last ambiguity cannot occur in Hardy fields, since in those the relation $\int h \succcurlyeq 1$ is determined.

Assume that $K$ is real-closed, and let $h \in K^{>}$. Then $h$ realizes $\boldsymbol{\lambda}$ in $K$ if and only if there is an H-field extension $K^{\star}$ of $K$ and a $g \in (K^{\star})^{>}$ which realizes $\boldsymbol{\gamma}$ in $K^{\star}$ with $h = -g^{\dagger}$ (in fact any such $g$ will realize $\boldsymbol{\gamma}$). Similarly, an element $h$ realizes $\boldsymbol{\omega}$ in $K$ if and only if there is an H-field extension $K^{\star}$ of $K$ and a $g \in (K^{\star})^{>}$ which realizes $\boldsymbol{\lambda}$ in $K^{\star}$ with $h = \omega(g)$. Now consider an ungrounded Hardy field $\mathcal{H}$. Then for $f \in \mathcal{H}^{>}$, we have

$$f \in \boldsymbol{\omega}_R \quad \text{iff all solutions of } 4\,y'' + f\,y = 0 \text{ generate a (common) Hardy field extension of } \mathcal{H},$$

$$f \in \boldsymbol{\omega}_L \quad \text{iff all non-trivial solutions of } 4\,y'' + f\,y = 0 \text{ oscillate.}$$

If $K$ is a real-closed H-field, then $h \in K^{>}$ realizes $\boldsymbol{\omega}$ if and only if some H-field extension has two linearly independent solutions of $4\,y'' + f\,y = 0$, and there is a differential field extension $K(y, y')$ of $K$ which cannot be ordered to make it an H-field extension of $K$.

There are other nice consequences of $\boldsymbol{\omega}$-freeness for real-closed H-fields with small derivation: that differentially algebraic H-field extensions remain $\boldsymbol{\omega}$-free, with mutually coinitial positive psi-sets.

### 4.3.3 The $\boldsymbol{\omega}$-function

Let $F$ be a differential field with $F = (F^{\times})^{\dagger}$ (e.g. $F$ is a Liouville-closed Hardy field). Consider a homogeneous linear ODE

$$y'' + a\,y' + b\,y = 0. \tag{4.3.1}$$

Let $g \in F^{\times}$ with $g^{\dagger} = -\frac{1}{2}\,a$, and set $z := y\,g^{-1}$. Then (4.3.1) is equivalent to

$$4\,z'' + f\,z = 0$$

for a certain $f \in F$. For $y \neq 0$, we have

$$\begin{aligned} 4\,y'' + f\,y &= y\left(\frac{4\,y''}{y} + f\right) \\ &= y\,(f - \omega(2\,y^{\dagger})). \end{aligned}$$

So $4\,y'' + f\,y$ has a non-trivial solution in $F$ if and only if $f = \omega(F)$. More details in [2, Section 5.2].

An H-field $K$ with asymptotic integration is $\boldsymbol{\omega}$-free if and only if it satifsifes

$$\forall f \in K, (\exists g \in K, ((g \succ 1) \wedge (f - \omega(g^{\dagger\dagger}) \succcurlyeq (g^{\dagger})^2))). \tag{4.3.2}$$

**Theorem 4.3.2.** *Every Hardy field extends differentially algebraically into an $\boldsymbol{\omega}$-free Hardy field.*

**08-03: Lecture 14**:

## 4.4 Differential algebra

### 4.4.1 Differential polynomials

Let $(R, \partial)$ be a differential ring (we impose in particular that $\mathbb{Q}$ is a subring of $R$). We write $R\{Y\}$ for the ring of differential polynomials with one indeterminate $Y$. As a ring this is $R[Y, Y', \ldots, Y^{(r)}, \ldots]$, but it is also an extension of $(R, \partial)$ where $\partial$ extends uniquely to $R\{Y\}$ by setting $\partial Y^{(r)} := Y^{(r+1)}$ for all $r \in \mathbb{N}$. For any differential ring extension $(S, \partial)$ of $R$ and $y \in S$, we have an evaluation map

$$\begin{array}{rcl} R\{Y\} & \longrightarrow & S \\ P & \longmapsto & P(y) \end{array}$$

which is the unique extension of differential rings sending $Y$ to $y$. We write $R\{y\}$ for the differential ring generated by $y$ over $R$, which is the range of the above map.

Inversely, each $P \in R\{Y\}$ can be seen as an operator $P(\partial): S \longrightarrow S$ which we often identify with $P$.

The order of $P \in R\{Y\}^{\neq}$, the *order* of $P$ is the least $r \in \mathbb{N}$ with $P \in R[Y, \ldots, Y^{(r)}]$. So differential polynomials of order 0 are just polynomials in $R[Y]$. Any $P \in R\{Y\}$ is the sum of its homogeneous parts $P = \sum_d P_d$ where $P_d$ is the homogeneous part of $P$ of degree $d$. The degree 1 part $P_1 = \sum_{i=0}^{r} a_i Y^{(r)}$ is particularly important.

### 4.4.2 Linear differential operators

The ring $R[\partial]$ is the ring of linear differential operators over $R$, which in general is non-commutative. It is free as a left $R$-module with basis $\{\partial^0, \partial^1, \ldots\}$. The product is given by composition of operators.

The product is given by extending the rules

$$\partial a := a \partial + a' \qquad \text{and} \qquad \partial \partial^r = \partial^{r+1}$$

for $a \in R$ and $r \in \mathbb{N}$. We see that $R[\partial]$ is commutative if and only if the derivation on $R$ is trivial. We also define $\operatorname{ord}(\sum_{i=0}^{r} a_i \partial^r) = r$ if $a_r \neq 0$, and we define the order of 0 to be $-\infty$. Each element $A$ of $R[\partial]$ acts as a $C_S$-linear operator on each differential ring extension $(S, \partial)$ of $R$, where $C_S$ is the constant ring of $(S, \partial)$. Composition of operators coincides with the product. In other words, we have an embedding $(R[\partial], +, \times) \longrightarrow (\operatorname{End}_{C_S}(S), +, \circ)$.

### 4.4.3 The case of differential fields

We now assume that $(K, \partial)$ is a differential field, with field of constants $C = \operatorname{Ker}(\partial)$. Then $K[\partial]$ has excellent algebraic properties: it is Euclidean in the expected sense (on the left *and* on the right).

For $A \in K[\partial]$, the space $\operatorname{Ker}_K(A) := \{y \in K : A(y) = 0\}$ is a finite dimensional subspace of the $C$-linear space $K$. In fact $\dim_C(\operatorname{Ker}_K(A)) \leqslant \operatorname{ord}(A)$.

**Factorization in $K[\partial]$.** Let $a \in K$. Then for $y \neq 0$ in a differential field extension $L$ of $K$, we have $(\partial - a)(y) = 0$ if and only if $a = y^{\dagger}$. So we have $A \in K[\partial]\,(\partial - a)$ if and only if we have $A(y) = 0$ for some $y \neq 0$ in some extension $L$ of $K$ with $y^{\dagger} = a$. Call $A \in K[\partial]$ irreducible if $\operatorname{ord}(A) > 0$ and $A \neq B\,D$ for all $B, D \in K[\partial]$ of order $> 0$. The Euclidean algorithm lets us write every $A$ as a product of irreducibles. The factorization is not unique (even up to units). We say that $A \in K[\partial]^{\neq}$ *splits* (over $K$) if $A = a\,(\partial - a_1) \cdots (\partial - a_r)$ for some $r \in \mathbb{N}$, $a \in K^{\times}$ and $a_1, \ldots, a_r \in K$. If $A \neq 0$ and $A = B\,D$, then $A$ splits over $K$ if and only if both $B$ and $D$ do.

**Returning to $K\{Y\}$**, we define additive and multiplicative conjugation. For $P \in K\{Y\}$ and $a \in K$, we define

$$\begin{aligned} P_{+a} &:= P(a + Y, a' + Y', \ldots) \\ P_{\times a} &:= P(a\,Y, a'\,Y + a\,Y', \ldots). \end{aligned}$$

The order and degree are preserved by those operations, (if $a \neq 0$ for multiplicative conjugation). We also have $P_{+a} = \sum_d (P_d)_{+a}$, so additive conjugation commutes with homogeneous parts.

### 4.4.4 Compositional conjugation

We still work within a differential field $(K, \partial)$. Let $\phi \in K^{\times}$. We consider the derivation $\partial_{\phi} := \phi^{-1}\,\partial$ on $K$. Rewriting $P \in K\{Y\}$ in terms of $\partial_{\phi}$ can sometimes drastically simplify things.

Let $K^{\phi}$ denote the differential field $(K, \partial_{\phi})$. So $K^1 = K$ and $(K^{\phi})^{\psi} = K^{\phi\psi}$ for all $\psi \in K^{\times}$. For $P \in K\{Y\}$, we claim that we have a differential polynomial $P^{\phi} \in K^{\phi}\{Y\}$ such that $P(y) = P^{\phi}(y)$ for all $y$ in $K$ and $K^{\phi}$ respectively (or even extensions thereof). Indeed, consider $\partial_{\phi}$ as the element $\phi^{-1}\,\partial$ of $K[\partial]$. We then have

$$\begin{aligned} \partial^2 &= \phi\,\partial_{\phi}\,\phi\,\partial_{\phi} \\ &= \phi\,(\partial_{\phi} + \phi^{-1}\,\phi')\,\partial_{\phi} \\ &= \phi^2\,\partial_{\phi}^2 + \phi'\,\partial_{\phi}, \\ \partial^3 &= \cdots \end{aligned}$$

and so on. We obtain $\partial^r = F^r(\phi)\,\partial_{\phi}^r + \cdots + F^1(\phi)\,\partial_{\phi}$ where each $F^i$ lies in $\mathbb{Z}\{Y\}$.

Using those identities, we define $P \mapsto P^{\phi}$ to be the unique $K$-algebra endomorphism of $K\{Y\}$ which sends each $a \in K$ to itself, $Y$ to itself, and each $Y^{(r)}, r > 0$ to $F^r(\phi)\,Y^{(r)} + \cdots + F^1(\phi)\,Y'$. Note that this operation is bijective, with inverse $Q \mapsto Q^{\phi^{-1}}$. Indeed, we have $(P^{\phi})^{\psi} = P^{\phi\psi}$ for all $P \in K\{Y\}$ and $\psi \in K^{\times}$. The compositional conjugation also preserves the degree and order, and commutes with homogeneous part and additive and multiplicative conjugations.

## 4.5 Eventual behavior

Now let $K$ be an H-field, with constant field $C$. We write $\mathcal{O} = K^{\preccurlyeq}$ and $\scriptstyle{\mathcal{O}} := K^{\prec}$. We also assume that $K$ has asymptotic integration. We consider the asymptotic couple $(\Gamma, \psi)$ of $K$. For $\phi \in K^{>}$, the field $K^{\phi}$ is still an H-field. The asymptotic couple of $K^{\phi}$ is $(\Gamma, \psi - v\phi)$, so its representation is just a vertical shift of that of $K$. The field $K^{\phi}$ has small derivation if and only if $v\phi \leqslant \gamma^{\dagger}$ for some $\gamma \in \Gamma^{\neq}$, i.e $v\phi \leqslant \rho$ for some $\rho \in \Psi$. When using compositional conjugations, we will only consider such $\phi$'s which satisfy this property, which we call *active*.

**Important phenomenon.** Let $P \in K\{Y\}^{\neq}$. As $v\phi$ increases, various quantities associated to $P^{\phi}$ stabilize. One such quantity is the so-called *dominant degree* of $P^{\phi}$.

### 4.5.1 Newton degree of a differential polynomial

Assume now that $K$ has small derivation. Then $\mathcal{O}$ and $\scriptstyle\mathcal{O}$ are differential subrings of $(K, \partial)$ (except for the fact that $\scriptstyle\mathcal{O}$ does not contain 1), so we have a natural differential ring mohphism $\mathcal{O}\{Y\} \longrightarrow C\{Y\}$ with kernel $\scriptstyle\mathcal{O}\{Y\}$, sending each $Y^{(r)}, r \in \mathbb{N}$ to itself and taking $a \in \mathcal{O}$ to its residue in $C$.

For $P \in K\{Y\}^{\neq}$, take an $a \in K^{\times}$ with $P = a\,Q$ where $Q \in \mathcal{O}\{X\} \setminus \scriptstyle\mathcal{O}\{Y\}$. Then $P \in a\,(D_P + o\{Y\})$ where $D_P$ is the image of $Q$ under $\mathcal{O}\{Y\} \longrightarrow C\{Y\}$. The number

$$\operatorname{ddeg} P := \deg D_P$$

does not depend on the choice of $a$ (and thus of $D_P$). We call it the *dominant degree* of $P$. We tend to take $a$ with valuation $va = \max\{v f_{\alpha} : \alpha \in \mathbb{N}^n\}$ where $P = \sum_{\alpha \in \mathbb{N}^n} f_{\alpha} Y^{(\alpha)}$. We define the *Newton degree* $\operatorname{ndeg} P$ of $P$ has the eventual value of $\operatorname{ddeg} P^{\phi}$ for active $\phi$ with sufficiently large valuation. We also set $\operatorname{ndeg} 0 := -\infty$. In fact even $D_{P^{\phi}}$ eventually stabilizes to a polynomial $N_P \in C\{Y\}$ called the *Newton polynomial* of $P$.

**Definition 4.5.1.** *We say that $K$ is* **Newtonian** *if every $P \in K\{Y\}^{\neq}$ with $\operatorname{ndeg} P = 1$ has a zero in $\mathcal{O}$.*

**Theorem 4.5.2.** *The field $K$ is $\boldsymbol{\omega}$-free if and only if for all $P \in K\{Y\}^{\neq}$, we have*

$$N_P \in K[Y]\,(Y')^{\mathbb{N}}.$$

**Theorem 4.5.3.** [ADH-PYNN-COATES] *Let $K$ be $\boldsymbol{\omega}$-free. Then $K$ is Newtonian if and only if it has no proper immediate d-algebraic H-field extension.*

**Theorem 4.5.4.** [ADH-PYNN-COATES] *Let $K$ be $\boldsymbol{\omega}$-free. Then there is an H-field extension $K^{Newt}/K$ where $K^{Newt}$ is Newtonian, and $K^{Newt}$ embeds (non-uniquely) into any other such expansion.*

The extension $K^{\mathrm{Newt}}/K$ is unique up to isomorphism over $K$. Besides, it is an immediate $d$-algebraic extension of $K$.

**Theorem A.** *The theory $T_{\mathrm{trans}}$ of $\boldsymbol{\omega}$-free, Newtonian, Liouville-closed H-fields with small derivation is complete.*

*The theory of $\boldsymbol{\omega}$-free, Newtonian, Liouville-closed H-fields is model complete and has two model completions (small der and not small der). This is the model companion of the theory of H-fields.*

**Theorem B.** *The field $\mathbb{T}$ of transseries is a model of $T_{\mathrm{trans}}$. Also the field $\mathbb{T}^{\mathrm{da}}$ of differentialy algebraic transseries is a model of $T_{\mathrm{trans}}$.*

# Chapter 5

# H-closed Hardy fields

## 5.1 Back to Hardy fields

Call a Hardy field $\mathcal{H}$ *d-maximal* if it has no proper $d$-algebraic extension which is a Hardy field. In particular, maximal Hardy fields are $d$-maximal. A Hardy field $\mathcal{H}$ is said H-closed if it is a model of $T_{\text{trans}}$, i.e. if it is $\boldsymbol{\omega}$-free, Newtonian and Liouville-closed.

**Theorem 5.1.1.** *A Hardy field $\mathcal{H}$ is d-maximal if and only if $\mathbb{R} \subseteq \mathcal{H}$ and $\mathcal{H}$ is H-closed.*

### 5.1.1 Some preleminary observations:

Let $\mathcal{H}$ be a Hardy field with asymptotic integration, and let $\phi \in \mathcal{H}^{>}$ be active. Then $\mathcal{H}^{\phi}$ is not a Hardy field in general, but it is isomorphic as an ordered, valued differential field to the Hardy field $\mathcal{H} \circ \psi^{\text{inv}}$ for any $\mathcal{H}$-Hardian germ $\psi$ with $\psi' = \phi$. Indeed, for $f \in \mathcal{H}$, we have $(f \circ \psi^{\text{inv}})' = \left(\frac{1}{\psi'} f'\right) \circ \psi^{\text{inv}}$, so

$$\begin{array}{rcl} \mathcal{H}^{\phi} & \longrightarrow & \mathcal{H} \circ \psi^{\text{inv}} \\ f & \longmapsto & f \circ \psi^{\text{inv}} \end{array}$$

is the desired isomorphism.

**Example 5.1.2.** Say that $\mathcal{H}$ is Liouville-closed, and take $\phi = \ell_n^{\dagger}$ (where $\ell_n$ is the $n$'th iterated log). Then $\phi = \exp_{n+1}$, so $\mathcal{H}^{\phi}$ is a Hardy field "with faster growing germs".

### 5.1.2 Trailing linear differential operators

For $a \in \mathbb{R}$, define $\mathcal{C}_a^r$ to be the $\mathbb{R}$-linear $\mathcal{C}^r$ functions $[a, +\infty) \longrightarrow \mathbb{R}$. We simply write $\mathcal{C}_a := \mathcal{C}_a^0$. Given $\phi \in \mathcal{C}_a^{r-1}$, we want to invert the $\mathbb{R}$-linear operator

$$\partial - \phi : \mathcal{C}_a^r \longrightarrow \mathcal{C}_a^{r-1}; f \mapsto f' - \phi f,$$

i.e. find a right inverse $B_{\phi}$ for $\partial - \phi$. We have

$$\begin{array}{rcl} B_{\phi} : \mathcal{C}_a^{r-1} & \longrightarrow & \mathcal{C}_a^r \\ g & \longmapsto & \left(t \mapsto \mathrm{e}^{\Phi(t)} \displaystyle\int_a^t \mathrm{e}^{-\Phi(s)} g(s) \, d s\right), \end{array}$$

where $\Phi(t) := \int_a^t \phi(s) \, d s$ for all $t \in [a, +\infty)$.

If $\phi \leqslant -\varepsilon$ on $[a, +\infty)$, then $B_{\phi}$ has good properties. Indeed, consider the space

$$\mathcal{C}_a^{r, \preccurlyeq} := \{f \in \mathcal{C}_a^r : f, f', \ldots, f^{(r)} \preccurlyeq 1\}.$$

This is a Banach space with norm $\|f\|_r := \max\{\|f\|, \|f'\|, \ldots, \|f^{(r)}\|\}$ where $\|\|$ is the sup norm on $\mathcal{C}_a$.

**Proposition 5.1.3.** *For $r \in \mathbb{N}^>$ and $\phi \in \mathcal{C}_a^{r-1,\preccurlyeq}$, the function $\partial - \phi: \mathcal{C}_a^{r,\preccurlyeq} \longrightarrow \mathcal{C}_a^{r-1,\preccurlyeq}$ is a continuous linear operator, and its operator norm is bounded. Likewise, the operator $B_\phi$ is continuous provided that $\phi \leqslant -\varepsilon$ on $[a, +\infty)$ for some $\varepsilon > 0$.*

The case when $\phi \leqslant -\varepsilon$ is called the *attractive case*. The opposite, "repulsive" case, i.e. when $\phi \geqslant \varepsilon$ for some $\varepsilon > 0$ on $[a, +\infty)$, then we need another right inverse

$$\begin{array}{rcl} B_\phi: \mathcal{C}_a^{r-1} & \longrightarrow & \mathcal{C}_a^r \\ g & \longmapsto & \left(t \mapsto \mathrm{e}^{\Phi(t)} \int_\infty^t \mathrm{e}^{-\Phi(s)}\, g(s)\, d\, s\right), \end{array}$$

for the same $\Phi$ as before. In that case, that $B_\phi$ is continuous.

We next need to consider the case which is neither attractive nor repulsive. Let $\mathfrak{m} \in (\mathcal{C}_a^1)^\times$ be a unit. Then $\mathfrak{m}\,(\partial - \phi)\,\mathfrak{m}^{-1} = \partial - (\phi + \mathfrak{m}^\dagger)$ and $\mathfrak{m}$ can sometimes be chosen so that $\phi + \mathfrak{m}^\dagger$ become attractive or repulsive. We then use compositional conjugation to work in something that is isomorphic to a Hardy field. Indeed for all units $\theta \in \mathcal{C}_a^\times$, we have

$$(Y' - \phi\, Y)^\theta = \theta\, Y' - \theta\, (Y' - \theta^{-1}\, \phi\, Y)$$

where choosing $\theta$ large enough, the function $\theta^{-1}\,\phi$ is in the attractive or repulsive case. The same works for complexifications, taking real parts for repulsive and attractive conditions.

**Remainder about smoothness of solutions of ODE's.** For $P \in R\{Y\}^{\neq}$ of order $r$ where $R$ is a differential ring, define

$$S_P := \frac{\partial P}{\partial Y^{(r)}}.$$

Note that $\deg_{Y^{(r)}} S_P < \deg_{Y^{(r)}} P$. Recall that $\mathcal{C}^\infty$ is a differential subring of $\mathcal{C}^{<\infty}$, and that $\mathcal{C}^\omega$ is a differential subring of $\mathcal{C}^\infty$. Let $P \in \mathcal{C}^{<\infty}\{Y\}^{\neq}$ have order $r$ and let $f \in \mathcal{C}^r$, so that $P(f)$ is defined. Suppose that $P(f) = 0$. If $S_P(f) \in \mathcal{C}^\times$, then $f \in \mathcal{C}^{<\infty}$. Similar results hold if $\mathcal{C}^{<\infty}$ is replaced by $\mathcal{C}^\infty$ or $\mathcal{C}^\omega$, or even in their complexifications.

A relevant special case: assume that $\mathcal{H}$ is a Hardy field and let $P \in \mathcal{H}\{Y\}$ be linear of order $r$ $P = Y^{(r)} + f_1\, Y^{(r-1)} + \cdots + f_r\, Y + R$ where $R \in o\{Y\}$. If $f \in \mathcal{C}^r$ is such that $P(f) = 0$ and that $f, f', \ldots, f^{(r)} \preccurlyeq 1$, then $S_P(f) \sim 1$ so $S_P(f) \in \mathcal{C}^\times$, whence $f \in \mathcal{C}^{<\infty}$. The same holds in the complexification.

Consider $f_1, \ldots, f_r \in \mathcal{C}_a^{\preccurlyeq}$ and the operator

$$A = \partial^r + f_1\, \partial^{r-1} + \cdots + f_r: \mathcal{C}_a^{r,\preccurlyeq} \longrightarrow \mathcal{C}_a^{\preccurlyeq}.$$

Assume that $A$ splits as a composition

$$A = (\partial - \phi_r) \cdots (\partial - \phi_1)$$

and we have for each factor $\partial - \phi_i, i \in \{1, \ldots, r\}$ a continuous right inverse $B_i: \mathcal{C}_a^{r-1,\preccurlyeq} \longrightarrow \mathcal{C}_a^{r,\preccurlyeq}$. Then we have a continuous right inverse $B_A = B_1 \cdots B_r$ for $A$.

### 5.1.3 Sketch of proof of the main theorem

We can now explain the proof of Theorem 5.1.1. We will call a Hardy field $\mathcal{H}$ $r$-Newtonian if it is "Newtonian for differential polynomials of order $r$".

**Very brief sketch of proof of Theorem 5.1.1.** We want to be able to construct a $d$-algebraic H-closed Hardy field extension of $\mathcal{H}$. We initially $d$-algebraically extend $\mathcal{H}$ into a Liouville-closed and $\boldsymbol{\omega}$-free Hardy field containing $\mathbb{R}$, and closed under $\cos, \sin : \mathcal{H}^{\preccurlyeq} \longrightarrow \mathcal{H}^{\preccurlyeq}$. By Zorn's lemma, it is enough to show that assuming that $\mathcal{H}$ is not Newtonian, it has a proper $d$-algebraic extension. By non-Newtonianity, there is a witness $(P, \mathfrak{m}, \hat{f})$ where $P \in \mathcal{H}\{Y\}^{\neq}$, $\mathfrak{m} \in \mathcal{H}^{\times}$ and $\hat{f}$ is a zero of $P$ which lies in an immediate H-field extension of $\mathcal{H}$, with $\hat{f} \notin \mathcal{H}$ and $\hat{f} \prec \mathfrak{m}$. We can choose this tuple to be lexicographically minimal for $(\operatorname{ord} P, \deg_{Y^{(r)}} P, \deg P) \in \mathbb{N}^3$. Since $\mathcal{H}$ is real-closed, we have $r := \operatorname{ord} P \geqslant 1$, and $\mathcal{H}$ is $(r-1)$-Newtonian. It follows that $\hat{f}$ is not the zero of a differential polynomial of order $< r$ over $\mathcal{H}$, so the extension $\mathcal{H}\langle \hat{f} \rangle$ is isomorphic over $\mathcal{H}$ to $\operatorname{Frac}(\mathcal{H}[Y, \ldots, Y^{(r)}]/(P))$ *via*

$$\begin{array}{rcl} \operatorname{Frac}(\mathcal{H}[Y, \ldots, Y^{(r)}]/(P)) & \longrightarrow & \mathcal{H}\langle \hat{f} \rangle \\ \dfrac{Q}{R} & \longmapsto & \dfrac{Q(\hat{f})}{R(\hat{f})}. \end{array}$$

We can also change $P$ without modifying the degrees to arrange that $\mathfrak{m} = 1$, so $\hat{f} \prec 1$: do this by taking $\left(P_{\times \mathfrak{m}}, 1, \hat{f}/\mathfrak{m}\right)$.

It is enough to find a germ $f \prec 1$ which is $\mathcal{H}$-Hardian such that $\mathcal{H}\langle f \rangle$ is isomorphic over $\mathcal{H}$, just as a field, to $\mathcal{H}\langle \hat{f} \rangle$. At a minimum, we want $f \in \mathcal{C}^{<\infty}$ such that $f \notin H$ and $P(f) = 0$. To that end, we use a fixed point construction. Let $A \in \mathcal{H}[\partial]$ be the linear differential operator corresponding to the homogeneous degree 1 part $P_1$ of $P$. One can arrange that $P_1$, and thus $A$, have order $r$. In order to make this sketch of proof possible, we make the **bold assumption** that $A$ splits over $\mathcal{H}$, i.e. that

$$A = h\,(\partial - \phi_1) \cdots (\partial - \phi_r)$$

for $h \in \mathcal{H}^{\times}$ and $\phi_1, \ldots, \phi_r \in \mathcal{H}$. By using other conjugations and tricks, we can arrange that $P \asymp 1$ and that $\phi_1, \ldots, \phi_r \asymp 1$. Pick representatives $f_j \in \mathcal{C}_a^r$ for each germ $\phi_j$, for a suitable common $a \in \mathbb{R}$. In fact since $\phi_1, \ldots, \phi_r \in \mathcal{H}$, we can impose that $f_j \in \mathcal{C}_a^{r,\preccurlyeq}$ by choosing $a$ large enough. Choosing $a$ even larger enougher, we impose that each factor $\partial - \phi_j$ is either in the attractive or repulsive case as per Section 5.1.2. This gives us a "good" right inverse $B$ of the geometric realization $A : \mathcal{C}_a^{r,\preccurlyeq} \longrightarrow \mathcal{C}_a^{\preccurlyeq}$ of $A$. Consider the (non-linear in general) operator

$$\begin{array}{rcl} \Psi : \mathcal{C}_a^{r,\preccurlyeq} & \longrightarrow & \mathcal{C}_a^{r,\preccurlyeq} \\ f & \longmapsto & B(P_1(f) - P(f)), \end{array}$$

and note that any fixed point of $\Psi$ is a zero of $P$. Indeed, assume that $f = B(P_1(f) - P(f))$. Then applying $A$ on both sides of the equality gives

$$P_1(f) = A(f) = P_1(f) - P(f),$$

hence the result. Recall that $\mathcal{C}_a^{r,\preccurlyeq}$ is a Banach space, so it is enough in order to prove that such a fixed point exists, to show that $\Psi$ is contractive on say $\mathcal{B} := \left\{ f \in \mathcal{C}_a^{r,\preccurlyeq} : \|f\|_r \leqslant \frac{1}{2} \right\}$. This can be done after transforming $(P, 1, \hat{f})$ into a "split normal form" through successive additive, multiplicative, compositional conjugations. More precisely, we arrange that $P = P_1 + R$ where $R$ is "tiny" compared to both $P$ and $P_1$. We have the liberty of increasing $a$ without changing the problem (e.g. $\|f\|_r \leqslant 1/2$ still holds), and then $\Psi$ is contractive and has fixed point $f \in \mathcal{C}_a^{r,\preccurlyeq}$ which is actually infinitesimal, not in $\mathcal{H}$, and also lies in $\mathcal{C}^{<\infty}$ by Remark 5.1.3.

We note three problematic issues:

1. The **bold assumption** might fail, and it is necessary to work over $\mathcal{H}[i]$ in general (which is a $d$-valued field). We also need to assume that $\mathcal{H}^{\preccurlyeq}$ is closed under cos and sin. All of this implies that we can assume that $\mathcal{H}[i]$ is $(r-1)$-Newtonian[5.1.1]. The complex or non-ordered version of $(r-1)$-Newtonianity is stronger and implies that all non-zero operators in $\mathcal{H}[i][\partial]$ split over $\mathcal{H}[i]$.
2. Then one must find a way to get back into the real valued case, starting from the solution $f \in \mathcal{C}^{<\infty}[i]$. Indeed write $f = g_1 + i\, g_2$ where $g_1, g_2 \in \mathcal{C}^{<\infty}$ and one of $g_1$ or $g_2$ is not in $\mathcal{H}$.
3. Even if one gets $f \in \mathcal{C}^{<\infty}$ with $f \prec 1$ and $P(f) = 0$, one still needs to show that $f$ is $\mathcal{H}$-Hardian.

As a final comment, the proof can be carried out in $\mathcal{C}^{\infty}$ instead of $\mathcal{C}^{<\infty}$. □

For the next few lines, we will focus on some aspects related to points 2 and 3 above, regarding exponential sums.

## 5.2 Exponential sums over Hardy fields

### 5.2.1 The universal exponential extension

Boshernitzan: Given a Hardian germ $f \in \mathcal{C}^{<\infty}$, we have the following equivalence:

$$f \succ \log \Longleftrightarrow f \text{ is uniformly distributed mod 1.}$$

Where $f$ being uniformly distributed mod 1 means that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \mathrm{e}^{2\pi i n f(t)}\, d\, t = 0$$

for each $n \in \mathbb{N}^{>}$.

Functions in the $\mathbb{R}$-linear span of $\{t \mapsto \mathrm{e}^{r i t} : r \in \mathbb{R}\}$ are called "amlost periodic". Using this and Boshernitzan's result, we can derive the following corrolary:

**Corollary 5.2.1.** *Let $\mathcal{H}$ be a Hardy field, let $\phi_1, \ldots, \phi_n \in \mathcal{H}$ such that*

$$\mathbb{R}\,\phi_1 + \cdots + \mathbb{R}\,\phi_n \cap \mathcal{H}^{\preccurlyeq} \subseteq \mathbb{R}.$$

*Then for all $f_1, \ldots, f_n \in \mathcal{H}[i]$, the germ of $f_1\, \mathrm{e}^{i\phi_1} + \cdots + f_n\, \mathrm{e}^{i\phi_n}$ is infinitesimal if and only if $f_1, \ldots, f_n$ are infinitesimal.*

Now let $\mathcal{H}$ be a Liouville-closed Hardy field containing $\mathbb{R}$, and assume that $K := \mathcal{H}[i]$ is 1-linearly surjective, i.e. all linear differential equations of order 1 over $K$ have solutions in $K$. For $f = g + h\, i \in K$ where $g, h \in \mathcal{H}$, we have $\mathrm{e}^f = \mathrm{e}^g\, (\cos(h) + \sin(h)\, i)$ where $\cos(h)$, $\sin(h) \in \mathcal{H}$ by 1-linear surjectivity. So $\mathrm{e}^f \in \mathcal{H} \cdot \mathrm{e}^{\mathcal{H} i} \subseteq \mathcal{C}^{<\infty}[i]$. Note that $(\mathrm{e}^f)' \subseteq K \cdot \mathrm{e}^{K i}$.

Write

$$\begin{aligned} \mathbb{U} &:= K[\mathrm{e}^K] \\ &= K[\mathrm{e}^{\mathcal{H} i}] \\ &= \{f_1\, \mathrm{e}^{i\phi_1} + \cdots + f_n\, \mathrm{e}^{i\phi_n} : f_1, \ldots, f_n \in \mathcal{H}[i] \wedge \phi_1, \ldots, \phi_n \in \mathcal{H}\}. \end{aligned}$$

5.1.1. this notion can be made sense of in the general, non-ordered context of $d$-valued fields or specifications thereof

So $\mathbb{U}$ is a differential subring of $\mathcal{C}^{<\infty}[i]$ extending $K$ and containing all constants (i.e. complex numbers). We call $\mathbb{U}$ the *universal exponential extension* of $K$. For $\phi \in \mathcal{H}$, we have $|e^{\phi i}| = 1$ and $(e^{\phi i})^{\dagger} = \phi'$. In fact $e^{\mathcal{H} i} = \{u \in \mathcal{C}^{<\infty} : |u| = 1 \wedge u^{\dagger} \in \mathcal{H}\}$. If moreover $\phi \preccurlyeq 1$, then $e^{\phi i} \in K$ already, so we are more interested in $\phi$'s which are positive infinite. We fix a decomposition $\mathcal{H} = \Lambda \oplus \mathcal{H}^{\preccurlyeq}$ into $\mathbb{R}$-linear spaces (think of $\Lambda$ as a space of purely infinite series). Note that $\Lambda \longrightarrow e^{\Lambda i}; \lambda \mapsto e^{\lambda i}$ is an isomorphism. We have that $\mathbb{U} = K[e^{\Lambda i}]$. This gives $\mathbb{U}$ the structure of a group ring over $K$, with $e^{\Lambda i}$ as the group.

**Proposition 5.2.2.** *The family* $(e^{\lambda i})_{\lambda \in \Lambda}$ *forms a basis of* $\mathbb{U}$ *over* $K$.

**Proof.** This family clearly generates $\mathbb{U}$ over $K$. Now assume for contradiction that $f_1 e^{i\lambda_1} + \cdots + f_n e^{i\lambda_n} = 0$ for some $n > 0$, $f_1, \ldots, f_n \in \mathcal{H}[i]^{\neq}$ and $\phi_1, \ldots, \phi_n \in \Lambda$. Then Corollary 5.2.1 implies that $f_1, \ldots, f_n$ are infinitesimal. But multipliying by a large $f \in \mathcal{H}$, we can assume that at least one $f_i$ is not infinitesimal: a contradiction. □

**Corollary 5.2.3.** *The ring* $\mathbb{U}$ *is a domain, with* $\mathbb{U}^{\times} = K^{\times}[e^{\Lambda i}]$.

Below, let $\lambda$ range in $\Lambda$. When using expressions like $\sum_{\lambda \in \Lambda} f_\lambda e^{\lambda i}$, we always assume that the family $(f_\lambda)_{\lambda \in \Lambda} \in K^{\Lambda}$ has finite support. Note that for $f = \sum_{\lambda \in \Lambda} f_\lambda e^{\lambda i} \in \mathbb{U}$, we have $\overline{f} = \sum_{\lambda \in \Lambda} \overline{f_\lambda} e^{-\lambda i}$, whence

$$f \in \mathcal{C}^{<\infty} \Longleftrightarrow \forall \lambda \in \Lambda, (f_{-\lambda} = \overline{f_\lambda}) \Longleftrightarrow \left( f \in \mathcal{H} \oplus \sum_{\lambda \in \Lambda^{>0}} \mathcal{H} \cos(\lambda) + \mathcal{H} \sin(\lambda) \right).$$

In particular, a basis of $\mathbb{U} \cap \mathcal{C}^{<\infty}$ as an $\mathbb{R}$-vector space is given by $(1, \cos(\lambda), \sin(\lambda))_{\lambda \in \Lambda^{>0}}$. We extend the valuation on $K$ to $\mathbb{U}$ by setting

$$v_g \sum_{\lambda \in \Lambda} f_\lambda e^{\lambda i} := \min \{v f_\lambda : \lambda \in \Lambda\}.$$

This is the unique valuation on $\mathbb{U}$ which extends the valuation on $\mathcal{H}$. We have a corresponding dominance relation denoted $\prec_g$ on $\mathbb{U}$. Note that for $f \in \mathbb{U}$, we have $f \prec_g 1$ if and only if $f_\lambda \prec 1$ for all $\lambda \in \Lambda$, whence, by Corollary 5.2.1, if and only if $f \prec 1$ in $\mathcal{C}^{<\infty}[i]$. In fact:

**Proposition 5.2.4.** *For* $f \in \mathbb{U}$, *and* $\mathfrak{m} \in K^{\times}$, *we have* $f \prec_g \mathfrak{m} \Longleftrightarrow f \prec \mathfrak{m}$ *and* $\mathfrak{m} \prec_g f \Longleftrightarrow \mathfrak{m} \prec f$. *The same holds for all other asymptotic relations* $\preccurlyeq_g$, $\sim_g$, *and so on…*

### 5.2.2 Exponential sums and linear differential operators

Let us show that each $A \in K[\partial]$ acts on $\mathbb{U}$ in a very transparent way. Let $A_\lambda := e^{-\lambda i} A e^{\lambda i} \in K[\partial]$. Then one can see that

$$A\left( \sum_{\lambda \in \Lambda} f_\lambda e^{\lambda i} \right) = \sum_{\lambda \in \Lambda} A_\lambda(f_\lambda) e^{\lambda i}.$$

Thus solving $A(y) = 0$ in $\mathbb{U}$ reduces to solving systems of equations $A_\lambda(y) = 0$ in $K$.

**Proposition 5.2.5.** *Let* $A \in K[\partial]^{\neq}$. *The* $\mathbb{C}$-*linear space* $\mathrm{Ker}_{\mathbb{U}}(A)$ *has a basis*

$$f_1 e^{\lambda_1 i}, \ldots, f_n e^{\lambda_n i}, \qquad \text{where } f_1, \ldots, f_n \in K^{\times},$$

*and for any such basis the* $f_i$*'s with* $\lambda_i = 0$ *form a basis of* $\mathrm{Ker}_K(A)$. *If* $A$ *morover splits over* $K$, *then* $n = \mathrm{ord}(A)$, *whence* $\mathrm{Ker}_{\mathbb{U}}(A) = \mathrm{Ker}_{\mathcal{C}^{<\infty}[i]}(A)$.

**Corollary 5.2.6.** *Assume that* $A\in\mathcal{H}[\partial]^{\neq}$*. The* $\mathbb{R}$*-linear space* $\mathrm{Ker}_{\mathbb{U}}(A)\cap\mathcal{C}^{<\infty}$ *has a basis*

$$g_1\cos(\lambda_1), g_1\sin(\lambda_1), \ldots, g_n\cos(\lambda_m), g_n\sin(\lambda_m), h_1, \ldots, h_n$$

*where* $g_1,\ldots,g_m\in\mathcal{H}^{\times}$*,* $\lambda_1,\ldots,\lambda_m\in\Lambda^{>0}$ *and* $h_1,\ldots,h_n\in\mathcal{H}^{\times}$*. For any such basis, the family* $(h_1,\ldots,h_n)$ *is a basis of* $\mathrm{Ker}_{\mathcal{H}}(A)$*. If moreover* $A$ *splits over* $K$*, then* $2m+n=\mathrm{ord}(A)$*, whence* $\mathrm{Ker}_{\mathbb{U}}(A)\cap\mathcal{C}^{<\infty}=\mathrm{Ker}_{\mathcal{C}^{<\infty}}(A)$*.*

# Chapter 6

# Filling gaps in Hardy fields

We now look into the proof of the following theorem

**Theorem 6.0.1.** *Let $\mathcal{H}$ be a Hardy field, let $L, R$ be countable subsets of $\mathcal{H}$ with $L < R$. Then there is an $\mathcal{H}$-Hardian germ $f \in \mathcal{C}^{<\infty}$ such that $L < f < R$.*

The special case $B = \varnothing$ was already treated by Sjödin-Arkiv-Mat in 1970. Hausdorff called a linear ordering $(X, <)$ an $\eta_1$-set if for all countable subsets $L, R \subseteq X$ with $L < R$, there is an $x \in X$ with $L < x < R$. So Theorem 6.0.1 is equivalent to the following:

**Corollary 6.0.2.** *Every maximal Hardy field is $\eta_1$ as a linear ordering.*

**Corollary 6.0.3.** *Any two maximal Hardy fields are back-and-forth equivalent.*

**Corollary 6.0.4.** *Assuming the continuum hypothesis, all maximal Hardy fields are isomorphic to the field $\mathbf{No}(\omega_1)$ of surreal numbers with countable birth day.*

It is unknown whether Theorem 6.0.1 holds in the analytic setting (but it does in the smooth setting).

## 6.1 Countable pseudo-Cauchy sequences

Let us start with a valuation theoretic characterization of $\eta_1$-ness in ordered valued fields. Let $K$ be an ordered field, with its natural valuation. Recall that its residue field $K^{\preccurlyeq}/K^{\prec}$ is Archimedean, hence it embeds uniquely into $\mathbb{R}$.

A sequence $(a_i)_{i\in\mathbb{N}}$ in $K$ is said to be *pseudo-Cauchy* (pc for short) if there exists a $i_0 \in \mathbb{N}$ such that for all $i, j, k \geqslant i_0$ with $i < j < k$, we have $a_k - a_j \prec a_j - a_i$. Equivalently, we have $a_{j+1} - a_j \prec a_{i+1} - a_i$ for all $j > i \geqslant i_0$.

Let $L$ be an ordered field extension of $K$. An element $a \in L$ is a *pseudo-limit* of a sequence $(a_i)_{i\in\mathbb{N}}$ if there is an $i_0 \in \mathbb{N}$ such that for all $j > i \geqslant i_0$, we have $a - a_j \prec a - a_i$. Note that this implies in particular that $(a_i)_{i\in\mathbb{N}}$ is pseudo-Cauchy. We then say that $(a_i)_{i\in\mathbb{N}}$ pseudo-converges (to $a$) and we write $(a_i)_{i\in\mathbb{N}} \rightsquigarrow a$.

**Example 6.1.1.** Take $K = \bigcup_{d>0} \mathbb{R}[[t^{\mathbb{Z}/d}]]$ as the field of formal Puiseux series, where $t > \mathbb{R}$. Then the sequence $(a_i)_{i\in\mathbb{N}}$ with $a_i := t + t^{1/2} + \cdots + t^{1/i+1}$ is pseudo-Cauchy, but does not pseudo-converge in $K$ itself.

**Lemma 6.1.2.** [4] *The two following conditions on $K$ are equivalent:*

*i. $(K, <)$ is $\eta_1$.*

ii. *The ordered residue field $K^{\preccurlyeq}/K^{\prec}$ is (isomorphic to) $\mathbb{R}$, every pc-sequence $(a_i)_{i\in\mathbb{N}}$ indexed by $\mathbb{N}$ has a pseudo-limit in $K$, and the value group of $K$ is $\eta_1$ as an ordered set.*

For (maximal) Hardy fields, the first part is a given, but maybe not the other two... We will focus on the pc-sequence part of the work.

Before we start, let us reformulate the problem of finding pseudo-limits in and out of ordered fields. Let $(a_i)_{i\in\mathbb{N}}$ be a pseudo-Cauchy sequence in $K$. All subsequences of $(a_i)_{i\in\mathbb{N}}$ are aso pseudo-Cauchy, and share limits in all ordered (and naturally valued) field extensions of $K$. By passing to a subsequence, we can arrange that $(a_i)_{i\in\mathbb{N}}$ is strictly monotonous, and given our $\eta_1$-ness problem, we might as well take opposites in the strictly decreasing case, hence imposing that $(a_i)_{i\in\mathbb{N}}$ is strictly increasing. Similarly, we may assume by translation and by taking a final segment of $(a_i)_{i\in\mathbb{N}}$ that: each $a_i$ is strictly positive (in particular $a_0$), and that $a_{i+1}-a_i \prec a_i - a_{i-1}$ for all $i>0$. Now for such a sequence $(a_i)_{i\in\mathbb{N}}$, we define $b_0 := a_0$ and $b_{i+1} := a_{i+1}-a_i$ for all $i\in\mathbb{N}$. Then we have

$$b_0 \succ b_1 \succ \cdots \qquad \text{and} \qquad b_0, b_1, \ldots > 0, \tag{6.1.1}$$

and $a_i := \sum_{k\leqslant i} b_i$ for all $i\in\mathbb{N}$. Using this, one can show that the following are equivalent:

i. All pc-sequences $(a_i)_{i\in\mathbb{N}}$ in $K$ pseudo-converge in $K$.

ii. For all $(b_i)_{i\in\mathbb{N}}$ satisfying (6.1.1), the pc-sequence $(b_0+\cdots+b_i)_{i\in\mathbb{N}}$ pseudo-converges in $K$.

## 6.2 Pseudo-limits in Hausdorff fields

Let $\mathcal{H}$ be a Hausdorff field containing $\mathbb{R}$ and let $f_i, i\in\mathbb{N}$ be strictly positive elements of $\mathcal{H}$ with $f_0 \succ f_1 \succ \cdots$. Set $F_n := f_0 + \cdots + f_n$ for each $n\in\mathbb{N}$. Let us try to construct a Hausdorff field $\mathcal{H}^* \supseteq \mathcal{H}$ which contains a pseudo-limit of $(F_n)_{n\in\mathbb{N}}$. Pick, by induction on $n\in\mathbb{N}$, a continuous representative $f_n\colon [1,+\infty) \longrightarrow \mathbb{R}$ of each germ $f_n$, such that $f_n(t)\geqslant 0$ and $f_{n+1}(t) \leqslant \frac{1}{2} f_n(t)$ for all $t\geqslant 1$. Thus, for each $t\geqslant 1$, the sum $F_n(t) := \sum_{n\in\mathbb{N}} f_n(t)$ is defined. The convergence of this series of functions is uniform on compact subsets of $[1,+\infty)$, hence $F$ is actually continuous on $[1,+\infty)$.

**Exercise 6.2.1.** For $n\in\mathbb{N}$, we have $F-F_n \prec f_n$ as germs.

**Lemma 6.2.1.** *If $\mathcal{H}$ is real-closed, and $(F_n)_{n\in\mathbb{N}}$ does not converge in $\mathcal{H}$, then $F$ generates an Hausdorff field extension $\mathcal{H}(F)$ of $\mathcal{H}$ with $(F_n)_{n\in\mathbb{N}} \rightsquigarrow F$. Moreover, the extension $\mathcal{H}(F)/\mathcal{H}$ is immediate.*

## 6.3 [A few missing notes]

[missing notes here, we take back after the proof of the main filling cuts result in the fluent case].

Assume now that $\left(\frac{f_{i+1}}{f_i}\right)_{i\in\mathbb{N}}$ is cofinal in $\mathcal{H}^{>,\prec}$. In particular, the cofinality of the psi set $\Psi$ is $\omega$. Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of positive active elements in $\mathcal{H}$ such that $(v\phi_n)_{n\in\mathbb{N}}$ is strictly increasing and cofinal in $\Psi$.

**Remark 6.3.1.** For any active $\phi > 0$ in $\mathcal{H}$, the arguments of last time give a germ $F_\phi \in \mathcal{C}^{<\infty}$ such that for all $k \leqslant m < \omega$, we have

$$(\phi^{-1}\partial)^{[k]}\left(\frac{\Phi_\phi - F_m}{f_m}\right) \prec 1. \tag{6.3.1}$$

However, this $\Phi_\phi$ depends on $\phi$. In particular, for each $n \in \mathbb{N}$, writing $\delta_n$ for the derivation $\delta_n := \phi_n^{-1}\partial$ on $\mathcal{C}^{<\infty}$, we obtain a $\Phi_n$ satisfying (6.3.1) above, with respect to $\delta_n$.

Let us construct a partition of unity $(\beta_n)_{n\in\mathbb{N}}$ such that $\Phi := \sum_{n\in\mathbb{N}} \beta_n \Phi_n$ exists and satisfies $\delta_n^{[k]}\left(\frac{\Phi - F_m}{f_m}\right) \prec 1$ for all $k \leqslant m < \omega$. Then one can show that for all active $\phi > 0$ in $\mathcal{H}$ and all $k \leqslant m < \omega$, we have

$$(\phi^{-1}\partial)^{[k]}\left(\frac{\Phi - F_m}{f_m}\right) \prec 1.$$

Then the key lemma implies that $\Phi$ is an $\mathcal{H}$-Hardian pseudo-limit of $(F_n)_{n\in\mathbb{N}}$. We choose $\beta_n$ as smooth functions that are zero outside of an interval $(a_n, b_{n+1})$, one on $(b_n, a_{n+1})$ increasing on $(a_n, b_n)$, decreasing on $(a_{n+1}, b_{n+1})$, and with $\beta_n + \beta_{n+1} = 1$ on $(a_{n+1}, b_{n+1})$. In fact we have a pointwise sum $\sum_{n\in\mathbb{N}} \beta_n = 1$ everywhere.

**Conjecture 6.3.2.** *Let $\mathcal{H}$ be an H-closed Hardy field, and let $y \in \mathcal{C}^{<\infty} \setminus \mathcal{H}$ be an $\mathcal{H}$-Hardian infinite germ. There is an $\varepsilon \in \mathcal{H}^>$ such that for all $f \in \mathcal{C}^{<\infty} \setminus \mathcal{H}$ with $f^{(n)} - y^{(n)} \prec \varepsilon^{(n)}$ for all $n \in \mathbb{N}$, the germ $f$ is also $\mathcal{H}$-Hardian, with a natural isomorphism $\mathcal{H}\langle y\rangle \longrightarrow \mathcal{H}\langle f\rangle$ over $\mathcal{H}$.*

**Question 5.** What about $y = \sqrt{x} + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}$? take $\varepsilon = \frac{1}{\mathrm{e}^x}$?????

## 6.4 Filling cuts in the value group

We now turn to the second part of the proof where we want to prove that the (underlying ordering of the) value group of a maximal Hardy field is an $\eta_1$-set.

**Theorem 6.4.1.** *Let $\mathcal{H}$ be a Liouville-closed Hardy field, and let $(\Gamma, \psi)$ denote its asymptotic couple. Suppose we have a $\beta$ in an H-asymptotic couple $(\Gamma^*, \psi^*)$ over $\mathbb{R}$ extending $(\Gamma, \psi)$ with the following properties:*

*i.* *$\beta \notin \Gamma$, and $\mathrm{cf}(\Gamma^{<\beta}) = \mathrm{ci}(\Gamma^{>\beta}) = \omega$, i.e. $\beta$ generates a countable cut in $\Gamma$.*

*ii.* *there are sequences $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ in $\Gamma$ and $\Gamma^*$ respectively such that $(\beta_i)_{i\in\mathbb{N}}$ is linearly independent modulo $\Gamma$, where $\beta_0 = \beta - \alpha_0$, $\beta_{i+1} = \beta_i^\dagger - \alpha_{i+1}$ for all $i \in \mathbb{N}$. So $\Gamma\langle\beta\rangle = \Gamma \oplus \bigoplus_{i\in\mathbb{N}} \mathbb{R}\beta_i$.*

*iii.* *$\beta_i^\dagger < 0$.*

*Then there is an $\mathcal{H}$-Hardian germ $y \in \mathcal{C}^{<\infty}$ such that $vy \in v\mathcal{H}\langle y\rangle$ realizes the same cut as $\beta$ in $\Gamma$.*

**Remark 6.4.2.** This deals in particular with the "nested monomial" case, taking $\alpha_i$ as the valuation of $\varphi_i$ and $\beta_i$ as the valuation of the successive nested monomials.

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# Glossary

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| $E(H)$ | intersection of all maximal Hardy fields containing $H$ | 14 |
| $\mathrm{ddeg}\, P$ | dominant degree of $P$ | 24 |
| $\mathrm{ndeg}\, P$ | Newton degree of $P$ | 24 |